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Research problems from the BCC22

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ABSTRACT

Open problems from the problem session at the 22nd British Combinatorial Conference at St Andrews, on 10 July 2009.

The Research Problems section presents unsolved problems in discrete mathematics. The problems here were posed at the problem session of the 22nd British Combinatorial Conference. Problems subsequently solved have been removed. The numbering is in the overall problem sequence for the journal, but the BCC numbers assigned in the first public version of the document have been included.

Technical comments and questions about a problem should be sent to its correspondent. Other information about partial or full solutions should be sent to Professor Cameron (for potential later updates). The problems are ordered by subject matter, beginning with block designs and Latin squares and moving on to graph theory and other topics.

PROBLEM 505. (BCC22.1) Nowhere-zero 5-flows for 2-designs

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A 2-*design* is a set of v *points* and a family of b subsets of these points called *blocks*, such that each block has size k and any two points lie in exactly λ common blocks, for some $\lambda > 0$. Let A be the v-by-b incidence matrix of a design. A *flow* is a vector in the null space of A; that is, $x \in \mathbb{R}^b$ satisfying $Ax = 0$. A flow is *nowhere-zero* if none of its entries are zero. It is known ([1]) that every 2-design with $b > v$ possesses a nowhere-zero flow.

Conjecture: Every 2-design with $b > v$ possesses a flow with all entries in $\{\pm 1, \pm 2, \pm 3, \pm 4\}$.

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PROBLEM 506. (BCC22.2) Infinite perfect Steiner triple systems

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Let S be a *Steiner triple system*: a family of blocks of size 3 on a set V such that every two points appear together in exactly one block. For $a, b \in V$, let $\{a, b, c\}$ be the unique block containing a and b, and let $V' = V \setminus \{a, b, c\}$. Define a graph $G_{a,b}$ with vertex set V' by making x adjacent to y if and only if $\{a, x, y\}$ or $\{b, x, y\}$ is a block. Since each element of V' lies in blocks with both a and b, the graph $G_{a,b}$ is 2-regular, and all cycles have even length. We say that S is *uniform* if the graphs $G_{a,b}$ are isomorphic, and S is *perfect* if each consists of a single cycle. Only finitely many finite perfect Steiner triple systems are known.

Question: Does there exist an infinite perfect Steiner triple system?

Comment: Such a system can be no larger than countably infinite, since an infinite "cycle" is a two-way infinite path and has only countably many vertices. Infinite uniform Steiner triple systems were constructed in [1], where each graph $G_{a,b}$ consists of infinitely many two-way infinite paths.

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PROBLEM 507. (BCC22.3) A- and D-optimality for block designs

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In general, a *block design* is a family of k-element subsets of a v-element set V. The *concurrence graph* of a design is the multigraph with vertex set V in which the number of edges with endpoints i and j is the number $\lambda_{i,j}$ of blocks containing elements i and j. The *Laplacian eigenvalues* of a design are the nonzero eigenvalues of the Laplacian matrix of the concurrence graph: for $i \neq j$, the (i, j) entry is $\lambda_{i,j}$, and the diagonal entries are chosen so that the row sums are zero. Thus there is a trivial eigenvalue 0 associated with the all-1 eigenvector (see also [2]).

In terms of the Laplacian eigenvalues, various optimality criteria for block designs are discussed in [1]. Specifically, A- and D-optimality involve respectively maximizing the harmonic mean and the geometric mean of the nontrivial eigenvalues. In [1], equivalent criteria are discussed. A block design is D-optimal (among designs in a given class) if its concurrence graph has the most spanning trees, and it is A-optimal if its concurrence graph maximizes the sum of resistances between pairs of terminals (when each edge is viewed as a 1-ohm resistor).

Block designs with $k = 2$ are just multigraphs. Fix $c \geq 0$. In [1], it is shown that, for block designs with $k = 2$ and $b - v = c$, there is a threshold v_0 such that if $v > v_0$, then the D-optimal designs are very different from the A-optimal designs.

Question 1: Does the conclusion of "very different" remain true when the family of designs is modified by
(1) Replacing $b = v + c$ with $b = cv$?
(2) Replacing $k = 2$ with general k?
(3) Replacing the quantity to be maximized in the A-criterion with $\left(\frac{1}{v-1} \sum_{i=1}^{v-1} \theta_i^{-p}\right)^{1/p}$ for some positive value p, where $\theta_1, \ldots, \theta_{v-1}$ are the nontrivial Laplacian eigenvalues? (If so, is v_0 a monotonic function of p?)

Problem 2: Quantify more precisely the meaning of "very different".

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PROBLEM 508. (BCC22.4) Squared Latin squares

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A *Latin square* of order n is an array containing n distinct symbols such that each symbol occurs once in each row and once in each column. A *subsquare* of a Latin square is a set of k rows and k columns such that the corresponding k^2 entries contain k distinct symbols and form a Latin square. The rows (or columns) of a subsquare need not be contiguous. It is known that the order of a proper subsquare is at most half the order of the full square.

Question: Does there exist a *squared Latin square*, that is, a Latin square partitioned into Latin subsquares, all of distinct sizes?

Comment: The freedom for the entries in Latin subsquares not to be contiguous in the full square is essential. A short argument due to Ian Wanless shows that no squared Latin square can be based on a "squared square" (i.e., a tiling of a square by smaller squares, all of distinct sizes; this geometric problem is discussed in [1]).

Douglas Stones observed that any squared Latin square must have order at least 21, using the fact that any proper Latin subsquare of a Latin square of order n must have order at most $n/2$. It seems likely that any example would in fact be significantly larger than this.

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PROBLEM 509. (BCC22.5) Which square has most autotopisms?

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An *autotopism* of a Latin square L is a triple (α, β, γ) of permutations of $\{1, \dots, n\}$ such that, if the (i, j) entry of L is k, then the (i^α, j^β) entry of L is k^γ, where i^α denotes $\alpha(i)$. The autotopisms of a Latin square form a group, the *autotopism group* of the square.

Question: Which Latin squares of given order n have the largest autotopism group? For which n is this maximum achieved by a group table?

Comment: R. A. Bailey showed that a group table does not always achieve the maximum. We represent a Latin square of order n by a collection of n^2 triples (a, b, c), where a, b and c are elements of a set S of size n; the entry in the row indexed by a and column indexed by b is c. Any two entries in a triple determine the third.

The Cayley table of a group G is a Latin square over the set G whose triples are (a, b, ab) for a, b in G. Schönhardt [3] showed that the autotopism group of such a Latin square is $(G \times G) \rtimes \mathrm{Aut}(G)$, generated by permutations of the form $(a, b, ab) \mapsto (ga, bh, gabh)$ for $g, h \in G$ and $(a, b, ab) \mapsto (a\alpha, b\alpha, (ab)\alpha)$ for $\alpha \in \mathrm{Aut}(G)$. Thus Cayley tables of groups with large automorphism groups yield Latin squares with large autotopism groups. Relative to their size, the groups with the largest automorphism groups are the elementary Abelian p-groups for prime p: if $|G| = p^m$, then $|\mathrm{Aut}(G)| = \prod_{i=0}^{m-1}(p^m - p^i)$.

Steiner triple systems also yield Latin squares. If S is the point-set of such a system, then the triples are (a, a, a) for a in S, and (a, b, c) (in all six orderings) whenever $\{a, b, c\}$ is a triple. The autotopism group of the corresponding square contains the group of automorphisms of the Steiner triple system, acting as quasigroup automorphisms (it may also be larger).

When $n = 2^m - 1$, the lines in projective $(m - 1)$-dimensional space over GF(2) form a Steiner triple system with automorphism group of order $\prod_{i=0}^{m-1}(2^m - 2^i)$. For example, when $n = 15$ this yields a Latin square of order 15 with autotopism group of order at least $15 \cdot 14 \cdot 12 \cdot 8$. The only group of order 15 is C_{15}, whose automorphism group has order 8, so the autotopism group of the Cayley table of C_{15} has order $15 \cdot 15 \cdot 8$, which is smaller. Thus when $n = 15$ the Latin square with the largest autotopism group is not a Cayley table. This may generalize to infinitely many numbers of the form $2^m - 1$.

A Latin square is *idempotent* if the diagonal entries are distinct; *unipotent* if they are identical. A Latin square made from a Steiner triple system of order n is idempotent, in that the diagonal triples all have the form (a, a, a). Such a square can be expanded to a unipotent Latin square of order $n + 1$ by adjoining ∞ to S, removing all triples of the form (a, a, a), inserting triples (∞, a, a), (a, ∞, a) and (a, a, ∞) for a in S, and inserting the triple (∞, ∞, ∞). The autotopism group of the new square is at least as large as that of the original.

Taking S to be the Steiner system whose triples are the lines in affine 4-dimensional space over GF(3), we obtain a Latin square of order 82 whose autotopism group has order at least $81 \cdot 80 \cdot 78 \cdot 72$. The only groups of order 82 are C_{82} and D_{82}, whose automorphism groups have orders 40 and $41 \cdot 40$, respectively. Since $81 \cdot 80 \cdot 78 \cdot 72 > 82 \cdot 82 \cdot 41 \cdot 40$, the Latin square of order 82 with the largest autotopism group is not a Cayley table. Again, this may hold for infinitely many numbers of the form $9^r + 1$.

If L and M are both Latin squares, then $\mathrm{Aut}(L \otimes M)$ contains $\mathrm{Aut}(L) \times \mathrm{Aut}(M)$. This observation yields a third class of examples. Taking L to be the square formed from the Steiner triple system defined by projective 4-dimensional space over GF(2) and taking M to have order 2 gives a Latin square of order 62 whose autotopism group has order at least $(31 \cdot 30 \cdot 28 \cdot 24 \cdot 16) \cdot (2 \cdot 2)$. The only groups of order 62 are C_{62} and D_{62}, whose automorphism groups have orders 30 and $31 \cdot 30$, respectively. Thus neither Cayley table has autotopism group as large as that of $L \otimes M$.

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PROBLEM 510. (BCC22.6) 2-coloured cycles in edge-colourings of complete bipartite graphs

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A Latin square of order n is equivalent to a proper n-edge-colouring of the complete bipartite graph $K_{n,n}$: if L has (i, j) entry k, we give colour k to the edge joining the ith vertex in the first partite set to the jth vertex in the second set. Since each colour appears at each vertex, the resulting colouring has cycles in two colours.

Let n be an integer greater than 1. Define $l(n)$ to be the minimum, over all proper n-edge-colourings of $K_{n,n}$, of the length of the longest 2-coloured cycle in such a colouring.

Question: Does $l(n) \leq 6$ hold for all sufficiently large n?

Comment: Cameron [1] proved that $l(n) = 4$ if and only if $n = 2^k$. Ninčák and Owens [2] proved that $l(n) \leq 2n - 4$ (with a few small exceptions). The upper bound was reduced to a polynomial in $\log n$ by Dukes and Ling [3], and it was subsequently reduced to the constant 1720 by the same authors [4].

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PROBLEM 511. (BCC22.7) Hemisystems in Hermitian varieties

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A *Hermitian variety* $H(2n + 1, q^2)$ is the set of isotropic points in the projective space $\mathrm{PG}(2n + 1, q^2)$ of a Hermitian form (a non-degenerate alternating bilinear form) on the underlying vector space $\mathrm{GF}(q)^{2n+2}$. A *generator* is a projective subspace of maximum possible dimension n contained in the Hermitian variety.

Question 1: When does a Hermitian variety $H(2n+1, q^2)$ possess a set G of generators such that, for some constant λ, every $(n - 1)$-dimensional space is incident with exactly λ elements of G?

Comment: The residue of any $(n - 2)$-dimensional space in $H(2n + 1, q^2)$ is isomorphic to $H(3, q^2)$. In this residue, such a set G would induce a set of lines covering every point λ times. A famous result by Segre [2] implies that q must be odd and that $\lambda = (q + 1)/2$.

Moreover, the residue of any $(n - 3)$-dimensional space in $H(2n + 1, q^2)$ is isomorphic to $H(5, q^2)$, where such a set G would induce a set of planes covering every line $(q + 1)/2$ times. Hence one might start by considering the following case of the original problem:

Question 1a: When does a Hermitian variety $H(5, q^2)$ possess a set G of planes such that every line is incident with exactly $(q + 1)/2$ planes of G?

Comment: No such sets of planes are known to exist in $H(5, q^2)$, although there are sets of planes for q odd that have the right size and cover every point the same number of times but do not cover lines the same number of times.

Problem 2: A *partial spread* is a set of pairwise disjoint generators. Prove in a geometric way that, if S is a partial spread in $H(4n + 1, q^2)$ with $|S| = q^{2n+1} + 1$, and if G is a set of generators covering every $(2n - 1)$-dimensional space exactly $(q + 1)/2$-times, then $|S \cap G| = |S|/2 = (q^{2n+1} + 1)/2$.

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PROBLEM 512. (BCC22.8) Spectral characterization of perfect matching in regular graphs?

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A *perfect matching* in a graph is a set of edges covering each vertex once. Various sufficient conditions for perfect matchings in regular or general graphs have been found in terms of the eigenvalues or Laplacian eigenvalues; see [2] for a survey of the area.

Question: Do there exist regular graphs G_1 and G_2 having the same spectrum such that G_1 has a perfect matching but G_2 does not?

Comment: A positive answer would show that the existence of a perfect matching is not determined by the spectrum of any of several variants of the adjacency matrix of the graph (such as the Laplacian or Seidel matrices).

Aidan Roy found a pair of non-regular graphs with the above property. The second is obtained from the first by Godsil–McKay switching [1].

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PROBLEM 513. (BCC22.9) Colouring vertex-transitive graphs

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Let G be a graph with n vertices having independence number $\alpha(G)$, chromatic number $\chi(G)$, and choosability (list chromatic number) $\mathrm{ch}(G)$. It is immediate that $\mathrm{ch}(G) \geq \chi(G) \geq n/\alpha(G)$, but how strong are these inequalities? By a famous result of Bollobás [3] and an extension due to Kahn (see Alon [1, Proposition 4.4]), for the standard Erdős–Renyi random graph with edge probability 1/2, with high probability all three quantities are $(n/(2\log_2(n)))(1 + o(1))$. That is, for any positive ϵ, with probability tending to 1 as $n \to \infty$ all three quantities are within a factor $1 + \epsilon$ of $n/(2\log_2(n))$.

Now we restrict our attention to vertex-transitive graphs and extremal (not expected) values. Always

$$\chi(G) \leq 1 + n\ln(n)/\alpha(G). \tag{1}$$

(See Babai [2], or the "Symmetric Hypergraph Theorem" in [5].)

Question 1: Does the same upper bound hold for $\mathrm{ch}(G)$ when G is vertex-transitive?

Comment: The proof of (1) in [2] considers random translates of a maximum independent set, assumed to be coloured with one colour; for choosability, we cannot assume the existence of such a set. When $G = K_{n/2,n/2}$, we have $n\ln(n)/\alpha = 2\ln(n)$, and $\mathrm{ch}(G) \simeq \log_2(n)$ for large n, so if (1) holds then the choosability bound cannot be improved by more than a small multiplicative constant. Although vertex-transitive graphs "look the same from all vertices", the lists involved in choosability very much need not, and hence the question may be difficult. Even if the specified bound does not hold, similar bounds would be of interest.

The *Kneser graph* $K(m, k)$ is the disjointness graph on the k-element subsets of $\{1, \ldots, m\}$; it is vertex-transitive, with $\binom{m}{k}$ vertices, independence number $\binom{m-1}{k-1}$ (by the Erdős–Ko–Rado Theorem [4]), and chromatic number $m - 2k + 2$ (by a result of Lovász [6]). Eleni Maistrelli (a student of Penman) observed that by letting $k = m/\omega(m)$ for ω tending to infinity (slowly) with m and simplifying, the Kneser graph shows that (1) cannot be improved much. Since Kneser graphs have "large" chromatic number, they may be a good candidate for showing that the bound of (1) does not hold for choosability.

Some classes of vertex-transitive graphs have chromatic number n/α (consider bipartite graphs or complete multipartite graphs, for example). This suggests other questions.

Problem 2: Find families of vertex-transitive graphs satisfying stronger upper bounds than (1); that is, prove theorems of the form "If G is a vertex-transitive graph with property P, then $\chi(G) \leq n\log(n)/(\alpha(G)f(n))$", where $f(n)$ is large. Alternatively, find further classes where (1) cannot be much improved.

Comment: Motivation comes from Gao's thesis (supervised by Penman), where (1) was used with results from Williford's thesis [8,7] to show that the Erdős–Rényi polarity graph ER_q has chromatic number at most $4\sqrt{2}\log(q)\sqrt{q}(1+o(1))$ when q is a power of 2 and $q \to \infty$. Since the lower bound n/α has order of magnitude $\sqrt{q}$, it would be interesting to see if the upper bound can be brought down to $c\sqrt{q}$ for some constant c. Answers to Question 2 could help.

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PROBLEM 514. (BCC22.10) Colourful paths

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Let G be a graph. With respect to a proper vertex colouring of G with $\chi(G)$ colours, a *colourful path* is a path with $\chi(G)$ vertices on which every colour appears. The concept was defined in [2]. If G is a connected graph, and v is a vertex of G, then there is a proper vertex colouring of G with $\chi(G)$ colours that contains a colourful path starting at v (see [1]).

Conjecture: Every connected graph G other than a 7-cycle has a proper vertex colouring with $\chi(G)$ colours such that, for every vertex v, there is a colourful path starting at v.

Comment: Li [3] (see also [4]) proved that for every optimal colouring of G and every vertex v in G, there is a path starting at v that represents all the colours, but the proof does not guarantee finding one with only $\chi(G)$ vertices.

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PROBLEM 515. (BCC22.11) Dynamic choosability

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A proper vertex colouring of a graph G is a *dynamic colouring* if on the neighbourhood of any vertex with degree at least 2 there appear at least two colours. The smallest number of colours in a dynamic colouring of G is the *dynamic chromatic number* of G, denoted $\chi_2(G)$.

A *list assignment* L assigns to every vertex v in a graph G a list $L(v)$ of available colours. An *L-colouring* is a proper colouring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. A graph G is *k-choosable* if it has an L-colouring whenever each assigned list has size at least k. The *list chromatic number* or *choosability* of G, denoted $\mathrm{ch}(G)$, is the least k such that G is k-choosable. A graph G is *dynamically k-choosable* if it has an L-colouring that is also a dynamic colouring whenever each assigned list has size at least k. The *dynamic choosability* of G, denoted $\mathrm{ch}_2(G)$, is the least k such that G is dynamically k-choosable.

The proposers initially conjectured that always $\mathrm{ch}_2(G) = \max(\mathrm{ch}(G), \chi_2(G)$. O. Riordan provided a counterexample. Begin with a triangle $\{a, b, c\}$. Take two copies of $K_{3,3}$; join all vertices of the first copy to b and all vertices of the second to c. Call this graph H. Now take two copies of H and a new vertex z that is joined to the two vertices labelled a in the two copies of H; the resulting graph is G. Now $\chi_2(G) = \mathrm{ch}(G) = 3$ but $\mathrm{ch}_2(G) > 3$. A document containing the proof is available from the editor.

Instead, a weaker alternative was proposed.

Question: Does there exist a function $f(x, y)$ such that always $\mathrm{ch}_2(G) \le f(\chi_2(G), \mathrm{ch}(G))$?

PROBLEM 516. (BCC22.12) Spanning trees of cubic graphs

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Say that a *green graph* is a cubic graph having a spanning tree in which all non-leaf vertices have degree 3. The Petersen graph, K_4, and all Halin graphs are green graphs, but the 8-vertex cube is not. We do not aim to characterize green graphs, but we conjecture that all connected cubic graphs have a nice property that for green graphs is immediate.

Conjecture 1: Every connected cubic graph G has a spanning tree whose leaves induce a 2-regular subgraph of G.

Comment: A solution to the following problem might be helpful in proving this conjecture.

Problem 2: Characterize the connected graphs with maximum degree 3 that have a matching whose deletion leaves a spanning tree.

PROBLEM 517. (BCC22.13) Synchronizing graphs

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A *unit representation* of a graph G is a function ρ that assigns to each vertex of G a point on the unit sphere in $\mathbb{R}^n$, for some n. The *energy* of ρ, written $\mathcal{E}(\rho)$, is defined by

$$\mathcal{E}(\rho) = \sum_{vw \in E} \|\rho(v) - \rho(w)\|^2.$$

We say that G is *synchronizing* if, for $n = 2$, every local minimum of $\mathcal{E}(\rho)$ is a global minimum.

Problem: Characterize the synchronizing graphs.

Comment: A solution would apply to coupled oscillators in dynamical systems; see [1].

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PROBLEM 518. (BCC22.15) Circuits in cubic bridgeless graphs

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These three problems share a theme but are successively more difficult. Let G be a cubic bridgeless graph. A *circuit* is a connected 2-regular graph.

Problem 1: Prove that G contains two circuits C and Q whose edge sets intersect in a matching M such that $G - M$ is bridgeless.

Problem 2: Prove that for every circuit C in G and every edge e in C, there is a circuit Q containing e that has the property specified in Problem 1: $E(C) \cap E(Q)$ is a matching M such that $G - M$ is bridgeless.

Problem 3: Prove that for every circuit C and every two consecutive edges e_1 and e_2 on C, there are circuits Q_1 and Q_2 such that
(1) $e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$;
(2) All of M_1, M_2, M_3 are matchings, where $M_1 = E(C) \cap E(Q_1)$, $M_2 = E(C) \cap E(Q_2)$ and $M_3 = E(Q_1) \cap E(Q_2)$; and
(3) $G - (M_1 \cup M_2 \cup M_3)$ is bridgeless.

PROBLEM 519. (BCC22.16) Domination polynomial

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A *dominating set* in a simple graph G is a set S of vertices of G such that every vertex not in S has a neighbour in S. The *domination polynomial* $D(G, x)$ is defined by

$$D(G, x) = \sum_{i=0}^{n} d(G, i)x^i,$$

where $d(G, i)$ is the number of dominating sets of size i in G, and n is the number of vertices of G. Graphs G and H are *D-equivalent* if $D(G, x) = D(H, x)$.

It is known that the D-equivalence class of the complete bipartite graph $K_{n,n}$ contains only $K_{n,n}$ and the cartesian product $K_n \Box K_2$. The D-equivalence class of $K_{n,n-1}$ also contains just two graphs, up to isomorphism.

Problem. Show that the D-equivalence class of $K_{m,n}$ consists only of $K_{m,n}$ if $m - n \geq 2$.

PROBLEM 520. (BCC22.17) Eating trees by cutting leaves

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Alice and Bob share a tree whose vertices have nonnegative weights. Starting with Alice, they alternately delete leaves (one in each move) and collect their weights. Both want to maximize their final gain.

There are trees on which the guaranteed outcome for Alice is zero, for example the path of length 2 with weights zero on both leaves and 1 on the intermediate vertex. If the tree has an even number of vertices, Alice can guarantee herself at least 1/4 of the total weight [7].

Conjecture: Alice can guarantee herself at least 1/2 of the total weight of any tree with an even number of vertices.

Comment: The fraction 1/2 is best possible.

Note added in proof: This conjecture has been proved by Deborah E. Seacrest and Tyler Seacrest in a short paper called "Grabbing the Gold" to appear in *Discrete Mathematics*.

References

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PROBLEM 521. (BCC22.18) Colouring intervals

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Consider the following game between two players Spoiler and Algorithm. Spoiler presents, one-by-one, unit-length intervals on the real line, in such a way that the maximum number of pairwise intersecting intervals (that is, the clique number of the corresponding interval graph) never exceeds w. Algorithm colours incoming intervals in such a way that intersecting intervals receive different colours.

Problem: How many colours does Algorithm require to perform the task?

Comment: It is known [1] that the number required is between $\lfloor 3w/2 \rfloor$ and $2w - 1$. The game on arbitrary intervals is studied in [2] and the answer to the analogous question in that setting is $3w - 2$. The problem is widely open also for the family of proper intervals (that do not nest).

References

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[2] H.A. Kierstead, W.T. Trotter, An extremal problem in recursive combinatorics, Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981), 1981, pp. 143–153.

PROBLEM 522. (BCC22.19) Universal graphs for spanning trees

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Question: Which subsets of edges can be removed from the complete graph K_n and still allow every unlabelled n-vertex tree to be embedded in what remains? In particular, do two maximal such sets always have the same size?

Comment: The maximum size of such edge sets was settled asymptotically by Chung and Graham in 1979 [1] (the proof is given in [2]). However, there seems to be no mention of the structural question.

References

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PROBLEM 523. (BCC22.20) Dispersed subsets of cycles

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For vertex subsets S and T in a graph G, let $d(S, T)$ denote the minimum distance between a vertex of S and a vertex of T that are not equal (note that S and T may be the same set).

Problem: Given positive integers $a, b, \alpha, \beta, \gamma$, what is the minimum length of a cycle containing vertex sets A and B of sizes a and b such that $d(A, A) \geq \alpha$, $d(B, B) \geq \beta$, and $d(A, B) \geq \gamma$?

Comment: With $L(a, b, \alpha, \beta, \gamma)$ denoting the minimum value, the presenter (unpublished) has shown that $L(a, a, 2, 2, \gamma) = \min(a\gamma, 4a + 2\gamma - 4)$.

PROBLEM 524. (BCC22.21) Abelian groups from triangulations

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Let T be a face 2-coloured triangulation in the plane with finitely many triangles, coloured black and white so that triangles sharing an edge have different colours. Associate an indeterminate x_v with each vertex v, and assume that these commute. Let G_W be the abelian group generated by the elements x_i, subject to the relations that the sum of the indeterminates on the vertices of each white triangle is zero. Define G_B similarly using the black triangles.

It is known (see Cavenagh and Wanless [1]) that the groups G_W and G_B have the same free rank, and their torsion subgroups have the same order.

Problem: Show that G_W and G_B are isomorphic.

References

[1] N. J. Cavenagh and I. M. Wanless, Latin trades in groups defined on planar triangulations, *J. Algebraic Combinatorics*, **30** (2009), 323–347.

PROBLEM 525. (BCC22.23) A non-invertibility graph

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Given a field F with characteristic other than 2, define a graph whose vertex set is the set of invertible $n \times n$ matrices over F, by putting matrices A and B adjacent if $A + B$ is singular.

It is known [1] that the clique number of this graph is finite.

Problem: Is the chromatic number of this graph finite?

References

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PROBLEM 526. (BCC22.24) Generating units by arithmetic progressions

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Let p be a prime, and fix $k \geq 3$. A *k-AP decomposition* of the group $\mathbb{U}_p$ of units of $\mathbb{Z}_p$ (the integers modulo p) is a k-tuple $(x_1, \ldots, x_k)$ of non-identity elements of $\mathbb{U}_p$ such that $(x_1, \ldots, x_k)$ is a k-term arithmetic progression in $\mathbb{Z}_p$, and $\mathbb{U}_p$ is the direct product of the cyclic groups generated by $x_1, \ldots, x_k$.

Many 3-AP decompositions exist; for example, $\mathbb{U}_{31} = \langle 25 \rangle \times \langle 30 \rangle \times \langle 4 \rangle$.

Problem: Do k-AP decompositions of $\mathbb{U}_p$ exist for $k > 3$?

Comment: No 4-AP decompositions exist for $p < 10,000$. We can also ask how the situation changes for composite moduli. For example, $\mathbb{U}_{104} = \langle 77 \rangle \times \langle 79 \rangle \times \langle 81 \rangle \times \langle 83 \rangle$.

PROBLEM 527. (BCC22.25) Sum-free subsets of $\mathbb{Z}_p$

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A subset A of $\mathbb{Z}_p$ is *sum-free* if $x, y \in A$ implies $x + y \notin A$. Let A be such a set. Lev [3] proved that if $|A| = m \leq 0.33p$, then there exists an element $k \in [1, \ldots, (p-1)/2]$ such that $kA \subseteq [m, \ldots, p-m]$ (where $kA = \{ka : a \in A\}$). Deshouillers and Freiman [1] and Deshouillers and Lev [2] proved that the statement remains true when 0.33 is replaced by 0.324 or 0.318, respectively. However, it is not true when 0.33 is replaced by 0.25.

Problem: Show that the statement remains true when 0.33 is replaced by any number strictly between 0.25 and 0.33.

References

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[2] J.-M. Deshouillers and V. F. Lev, Refined bound for sum-free sets in groups of prime order, *J. London Math. Soc.* **40** (2008), 863–875.
[3] V. F. Lev, Large sum-free sets in Z/Z_p, *Israel J. Math.* **154** (2006), 221–234.

PROBLEM 528. Determinant and permanent

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The *permanent* of a square matrix is the sum of all the products occurring in the expansion of the determinant, but without the signs affixed to them in the determinant expansion. For example, the permanent of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is 2.

Question: Among the $n \times n$ matrices A of zeros and ones that satisfy $|\det(A)| = \mathrm{per}(A)$, what is the maximum value of $|\det(A)|$?

Comment: For $1 \leq n \leq 8$, the maxima are 1, 1, 2, 3, 5, 8, 24, 24. An example of a 7×7 matrix with determinant and permanent 24 is the incidence matrix of the *Fano plane* (the unique Steiner triple system with 7 points). See [1] for more discussion.

References

[1] N. J. Cavenagh and I. M. Wanless, Latin trades in groups defined on planar triangulations, *J. Algebraic Combinatorics*, **30** (2009), 323–347.